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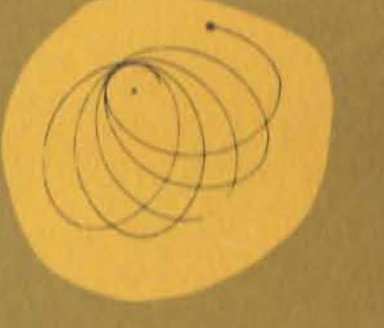

ALMOST

PERIODIC

FUNCTIONS

A. S. Besicovitch

ALMOST PERIODIC FUNCTIONS

by A. S. Besicovitch

This important summary by a well-known mathematician covers the theory of almost periodic functions created by Harold Bohr. It examines Bohr's own work, as well as newer, shorter, and more elementary proofs than Bohr's, and also demonstrates extensions of the theory beyond the class of uniformly continuous functions to which Bohr's work was limited. The contributions of Wiener, Weyl, de la Vallée Poussin, Stepanoff, and Bochner are examined, while the author's own work on the piecewise continuous case is also included.

The first portion of this book establishes basic theorems of uniformly a.p. functions, including Bohr's original work, and de la Vallée Poussin's ingenious short proof based on classical theory of purely periodic functions. It considers such matters as summation of Fourier series of u.a.p. functions by partial sums, the Bochner-Fejer summation of u.a.p. functions, particular cases of Fourier series, and u.a.p. functions of two variables.

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CONTENTS

PREFACE

THE theory of almost periodic functions, created by H. Bohr, has now completed two stages of its development.

Almost periodicity, as a structural property, is a generalisation of pure periodicity and Bohr's original methods for establishing the fundamental results of the theory were always based on reducing the problem to a problem of purely periodic functions. But though the underlying idea of Bohr's method was clear and simple, the actual proofs of the main results were very difficult and complicated. New methods were given by N. Wiener and H. Weyl, by which the results were arrived at in a much shorter way. But these methods have lost the elementary character of Bohr's methods. It was C. de la Vallée Poussin who succeeded in giving a new proof (based partly on H. Weyl's idea), which was very short and at the same time based entirely on classical results in the theory of purely periodic functions.

This represents one stage in the development of the theory of almost periodic functions.

Bohr's theory of *a.p.* functions was restricted to the class of uniformly continuous functions. Then efforts were directed to generalisations of the theory. Thanks to the work of W. Stepanoff, N. Wiener, H. Weyl, H. Bohr and others, generalisations may be considered to have reached a certain completeness.

This was the second stage in the development of the theory.

These circumstances suggest that the present moment is not unfavourable for writing an account of the theory.

In Chapter I of this account we develop the fundamental part of the theory of *a.p.* functions of a real variable—the theory of uniformly *a.p.* functions. In the main problems of this chapter we adopt the methods of H. Bohr, de la Vallée Poussin, Weyl and Bochner.

Chapter II is devoted to a systematic investigation of generalisations of the theory.

In Chapter III we develop the theory of analytic *a.p.* functions, which in essentials remains unaltered as it was published by H. Bohr.

This account is not encyclopaedic. Our aim is to give the fundamental results of the theory, and we have omitted all discussion of certain special problems. Thus the work of Bohr, Neugebauer, Walter and Bochner on differential equations and difference equations has not been considered in this book, nor has the theory of harmonic *a.p.* functions developed by J. Favard in his interesting paper. For all these questions the reader is referred to the original papers.

A. S. B.

INTRODUCTION

THE theory of almost periodic functions was created and developed in its main features by H. Bohr during the last decade. Like many other important mathematical discoveries it is connected with several branches of the modern theory of functions. On the one hand, almost periodicity as a structural property of functions is a generalisation of pure periodicity: on the other hand, the theory of almost periodic functions opens a way of studying a wide class of trigonometric series of the general type and of exponential series (Dirichlet series), giving in the latter case important contributions to the general problems of the theory of analytic functions.

Almost periodicity is a generalisation of pure periodicity: the general property can be illustrated by means of the particular example

$$f(x) = \sin 2\pi x + \sin 2\pi x \sqrt{2}.$$

This function is not periodic: there exists no value of τ which satisfies the equation

$$f(x+\tau) = f(x)$$

for all values of x. But we can establish the existence of numbers for which this equation is approximately satisfied with an arbitrary degree of accuracy. For given any $\epsilon > 0$ as small as we please we can always find an integer τ such that $\tau\sqrt{2}$ differs from another integer by less than $\epsilon/2\pi$. It can be proved that there exist infinitely many such numbers τ, and that the difference between two consecutive ones is bounded. For each of these numbers we have

$$\begin{aligned} f(x+\tau) &= \sin 2\pi(x+\tau) + \sin 2\pi(x+\tau)\sqrt{2} \\ &= \sin 2\pi x + \sin(2\pi x\sqrt{2} + \theta\epsilon) \qquad (|\theta| \leq 1) \\ &= f(x) + \theta'\epsilon. \qquad (|\theta'| \leq 1) \end{aligned}$$

Almost periodicity of a function $f(x)$ in general is defined by this property:

The equation

$$f(x+\tau)=f(x)$$

is satisfied with an arbitrary degree of accuracy by infinitely many values of τ, these values being spread over the whole range from $-\infty$ to $+\infty$ in such a way as not to leave empty intervals of arbitrarily great length.

Almost periodicity is a deep structural property of functions which is invariant with respect to the operations of addition (subtraction) and multiplication, and also in many cases with respect to division, differentiation, integration and other limiting processes.

To the structural affinity between almost periodic functions and purely periodic functions may be added an analytical similarity. To any almost periodic function corresponds a "Fourier series" of the type of a general trigonometric series

$$(1) \qquad f(x) \sim \sum_{n=1}^{\infty} A_n e^{i\Lambda_n x}$$

(Λ_n being real numbers and A_n real or complex): it is obtained from the function by the same formal process as in the case of purely periodic functions (namely, by the method of undetermined coefficients and term-by-term integration). The series (1) need not converge to $f(x)$, but there is a much closer connection between the series and the function than we have yet seen. In the first place Parseval's equation is true, i.e.

$$M\{|f(x)|^2\} = \Sigma\, |A_n|^2,$$

from which follows at once the uniqueness theorem, according to which there exists at most one almost periodic function having a given trigonometric series for its Fourier series. Parseval's equation constitutes the fundamental theorem of the theory of almost periodic functions. Further, the series (1) is "summable to $f(x)$," in the sense that there exists a sequence of polynomials

$$\sum_{n=1}^{\infty} p_n^{(k)} A_n e^{i\Lambda_n x} \qquad (k=1, 2, \ldots)$$

(where $0 \leqq p_n \leqq 1$, and where for each k only a finite number of the factors p differ from zero) which

(*a*) converge to $f(x)$ uniformly in x, and

(*b*) converge formally to the series (1),

by which is meant that for each n

$$p_n^{(k)} \to 1, \quad \text{as} \quad k \to \infty.$$

Conversely, any trigonometric polynomial is an almost periodic function, and so is the uniform limit of a sequence of trigonometric polynomials. It is easily proved that the Fourier series of such a limit function is the formal limit of the sequence of trigonometric polynomials. Thus the class of Fourier series of almost periodic functions consists of all trigonometric series of the general type

$$\Sigma A_n e^{i\Lambda_n x},$$

to which correspond uniformly convergent sequences of polynomials of the type

$$\Sigma p_n^{(k)} A_n e^{i\Lambda_n x}, \qquad (k=1, 2, \ldots)$$

formally convergent to the series. Thus the theory of almost periodic functions opened up for study a class of general trigonometric series: the extent of this class will be discussed later on.

The first investigations of trigonometric series other than purely periodic ones were carried out by Bohl. He considered the class of functions represented by series of the form

$$\sum_{(n)} A_{n_1, n_2, \ldots, n_l} e^{i(n_1 a_1 + n_2 a_2 + \ldots + n_l a_l)x},$$

where $a_1, a_2, \ldots, a_l$ are arbitrary real numbers, and $A_{n_1, n_2, \ldots, n_l}$ real or complex numbers. The necessary and sufficient conditions that a function is so representable are that it possesses certain quasi-periodic properties which are at first glance very similar to almost periodicity; but Bohl's restriction on the exponents of the trigonometric series places his problem in the class of those whose solution follows in a more or less natural way from existing theories rather than of those giving rise to an entirely new theory.

A quite new way of studying trigonometric series is opened up by Bohr's theory of almost periodic functions. We indicated above the class of trigonometric series which correspond to almost periodic functions: we have not yet indicated how wide the class is. It is not possible to give any direct test for a series to be the Fourier series of an almost periodic function, nor can the similar problem be solved for the class of purely periodic continuous functions. But when the property of almost periodicity is properly generalised, then the corresponding class of Fourier series acquires a rather definite character of completeness.

The original work of Bohr was confined to the almost periodicity defined above. Thereafter work was done in the way of generalisation of the property by Stepanoff, Wiener, Weyl, Bohr and others. The new types of almost periodic functions were represented by new classes of trigonometric series

$$\Sigma A_n e^{i\Lambda_n x}.$$

As before, to a series of one of the new classes still corresponds a convergent series of polynomials

$$\Sigma p_n^{(k)} A_n e^{i\Lambda_n x}, \qquad (k = 1, 2, \ldots)$$

but to each new type of almost periodicity corresponds a different kind of convergence of this sequence—not uniform convergence, although the convergence always has some features of uniformity. In fact there exists a strict reciprocity between the kind of almost periodicity of a function and the kind of convergence of the corresponding sequence of polynomials.

When these generalisations are taken into consideration some answer can be made to the above question of the extent of the class of trigonometric series which are Fourier series of almost periodic functions. This answer is given by the Riesc-Fischer Theorem:

Any trigonometric series $\Sigma A_n e^{i\Lambda_n x}$, subject to the single condition that the series $\Sigma |A_n|^2$ is convergent, is the Fourier series of an almost periodic function.

This is completely analogous to the Riesc-Fischer Theorem for the case of purely periodic functions. Generalisations of this result similar to those for purely periodic functions are possible.

Thus the Fourier series of all almost periodic functions form as large a subset of the class of all trigonometric series of the general type as the Fourier series of purely periodic functions do of the class of all trigonometric series of the ordinary type.

There is no doubt whatever that a trigonometric series of the general type

$$\Sigma A_n e^{i\Lambda_n x}$$

(with no restriction on the coefficients), in general does not represent a function (is not "summable") in any natural way, and it may be that almost periodicity is the decisive test for a non-artificial summability.

Almost periodicity is generalised in a natural way to the class of analytic functions in a strip $a < Rz < b$ by the condition that the approximate equation

$$f(z + \tau i) = f(z), \qquad (\tau \text{ real})$$

must be satisfied in the whole strip. One of the main features of the theory of analytic almost periodic functions is the existence of the "Dirichlet series" $\Sigma A_n e^{\Lambda_n z}$, which corresponds to the Fourier series of almost periodic functions of a real variable. The consequence is the same as in the case of almost periodic functions of a real variable. We get a possibility of enlarging the class of exponential series accessible to investigation. While in the case of ordinary Dirichlet series $\Sigma A_n e^{\Lambda_n z}$ the exponents are subject to the condition of forming a monotone sequence, there is no restriction of this kind on Dirichlet series of almost periodic functions. In fact any set of real numbers may form Dirichlet exponents of an analytic almost periodic function.

The connection between analytic functions and their Dirichlet series is even deeper than between almost periodic functions of a real variable and their Fourier series. The behaviour of almost periodic functions and the character of their singularities at infinity are defined by the nature of their Dirichlet series.

The study of harmonic and doubly periodic functions has also brought interesting and important results.

Applications of almost periodic functions have been made to linear differential and difference equations, and undoubtedly further development of the theory will lead to wider applications.